Exam Calculus of Variations and Optimal Control 2019-20

Date	:	24-01-2020
Place	:	Martiniplaza
Time	:	08.30-11.30

The exam is OPEN BOOK; you can use all your books/papers/notes; but NO internet connection.
You are supposed to provide arguments to all your answers, and to explicitly refer to theorems/propositions whenever you use them.

1. Consider the minimization of

$$J(x(\cdot)) = \int_0^1 \left(\dot{x}^2(t) + x^2(t) + 2tx(t)\right) dt$$

over all functions $x : [0,1] \to \mathbb{R}$ with begin and end condition

$$x(0) = 0, \quad x(1) = 1$$

   (a) Determine the Euler-Lagrange equation for this problem.
   (b) Is Legendre's condition (see Theorem 1.7.1) satisfied ?
   (c) Let $x_*(\cdot)$ be the solution of the Euler-Lagrange equation. Prove that it is a global minimum of $J(x(\cdot)) = \int_0^1 \left(\dot{x}^2(t) + x^2(t) + 2tx(t)\right) dt$ by using Theorem 1.7.6.
   (d) Compute the solution $x_*(\cdot)$ be of the Euler-Lagrange equation. Show that

$$J(x_*(\cdot) + \delta_x(\cdot)) = J(x_*(\cdot)) + \int_0^1 \left(\delta_x^2(t) + \dot{\delta}_x^2(t)\right) dt$$

   for every continuously differentiable variation function $\delta_x(\cdot)$ with $\delta_x(0) = \delta_x(1) = 0$. Conclude from here, without using Theorem 1.7.6, that $x_*(\cdot)$ is a global minimum.

2. Consider the scalar system $\dot{x} = u$.

   (a) Determine the optimal feedback $u = -fx$ (f a scalar to be computed) for minimizing the infinite horizon LQ-problem

$$\int_1^\infty (u^2(t) + 4x^2(t))dt$$

   with given initial condition $x(1) = x_1$, and show that the minimal cost (as a function of x_1) is given as $V(x_1) = 2x_1^2$.

(b) Determine by solving the Riccati differential equation the optimal input function $u : [0, 1] \to \mathbb{R}$ minimizing

$$\int_0^1 2u^2(t)dt + kx^2(1)$$

for given constant k, with given initial condition $x(0) = x_0$. Determine the minimal costs as a function of x_0.

(c) Determine on the basis of parts (a) and (b) the optimal control $u^* : [0, \infty) \to \mathbb{R}$ for minimizing the cost criterion

$$\int_0^1 2u^2(t)dt + \int_1^\infty (u^2(t) + 4x^2(t))dt \tag{1}$$

with given initial condition $x(0) = x_0$. Argue carefully that the restriction of the optimal input function $u^* : [0, \infty) \to \mathbb{R}$ to the interval $[1, \infty)$ is the solution to part (a) for $x(1) = x^*(1)$, where $x^*(1)$ is the state at time 1 resulting from applying the optimal control found in part (b).

Determine the total minimal cost of (1) as a function of x_0.

3. Consider the system (with e of course the well-known number $e = 2.71..$)

$$\dot{x} = x(1-u), \quad x(0) = 1, \; x(1) = \frac{e}{2}$$

with cost

$$\int_0^1 -\ln\left(x(t)u(t)\right) dt$$

(a) Determine the Hamiltonian, and the Hamiltonian equations (2.11) with boundary conditions.

(b) Show that

$$u(t) = -\frac{1}{p(t)x(t)}$$

is the candidate optimal control as a function of state $x(t)$ and co-state $p(t)$.

(c) Use the previous part in order to compute the optimal $p_*(\cdot)$, $x_*(\cdot)$, and eventually $u_*(\cdot)$.

(d) Show that $x_*(t) > 0$, and $u_*(t) > 0$ for all $t \in [0, 1]$ (and thus that the cost is well-defined).

(e) Derive the Hamilton-Jacobi-Bellman equation for this optimal problem.

4. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1 \\ \dot{x}_2 &= -x_1^3 \end{aligned}$$

with equilibrium $(0, 0)$.

(a) Can you conclude anything about the stability of $(0, 0)$ by linearization ?

(b) Show that

$$V(x_1, x_2) = x_1^4 + 2x_2^2$$

is a Lyapunov function, and conclude stability.

(c) Is $(0, 0)$ asymptotically stable, and if so, how do you prove this ? What about global asymptotic stability ?

Distribution of points: Total 100; Free 10.

1. a: 5, b: 5, c: 5, d: 5
2. a: 10, b: 10, c: 10
3. a: 5, b: 3, c: 5, d: 5, e: 2
4. a: 5, b: 5, c: 10